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NOTE ON THE CONVERGENCE OF A SEQUENCE OF FUNCTIONS OF A CERTAIN TYPE

BY H. E. BUCHANAN AND T. H. HILDEBRANDT

THEOREMS concerning uniform convergence of sequences and series of functions are numerous. Perhaps as well known as any is:

"If a sequence of continuous functions converges uniformly, the limit function is continuous."

The converse of this theorem does not, in general, hold. Harnack* proves a converse theorem for the case where the sequence of functions $f_n(x)$ is such that, for a definite n' and for every x, if $n \geqq n'$,

$$f_n(x) \leqq f_{n+1}(x).$$

Osgood† states a converse for the case

$$|f_n(x') - f_n(x)| \leqq M|x' - x|,$$

where M is a fixed positive quantity and x' and x are any two points of the interval considered. A converse of a slightly more general type than those of Harnack and Osgood is the following:

THEOREM A. *If a sequence of monotonic nondecreasing‡ functions* $f_n(x)$, $(n = 1, 2, 3, \cdots)$, *converges to a function* $f(x)$, *continuous on the interval* $ab (a \leqq x \leqq b)$, *then* $f(x)$ *is a monotonic nondecreasing function, and the convergence is uniform over the interval* ab.

1. *The limit function is monotonic nondecreasing.* Suppose it were not. Then for certain two points $x_1, x_2 (a \leqq x_1 < x_2 \leqq b)$ we would have

$$f(x_1) > f(x_2).$$

Let
$$f(x_1) - f(x_2) = h.$$

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* *Differential-und Integralrechnung*, p. 234.

† ANNALS OF MATHEMATICS, ser. 2, vol. 3, p. 139, 1902.

‡ Of course the theorem holds if everywhere "nondecreasing" is replaced by "nonincreasing."

Since $f_n(x)$ converges for all values of x on the interval $a \leqq x \leqq b$, we have

$$\underset{n}{L} f_n(x_1) = f(x_1)\,; \quad \underset{n}{L} f_n(x_2) = f(x_2)$$

that is, if there is given an $\epsilon > 0$, it is possible to find an $n_{\epsilon x_1}$ of such a nature that if $n \geqq n_{\epsilon x_1}$, we have

$$| f_n(x_1) - f(x_1) | \leqq \epsilon\,;$$

and an $n_{\epsilon x_2}$ of such a nature, that if $n \geqq n_{\epsilon x_2}$ we have

$$| f_n(x_2) - f(x_2) | \leqq \epsilon.$$

Suppose $\epsilon < \frac{h}{2}$ and choose n greater than or equal to $n_{\epsilon x_1}$ and $n_{\epsilon x_2}$. Then

$$| f_n(x_1) - f(x_1) | < \tfrac{1}{2}h \quad \text{and} \quad | f_n(x_2) - f(x_2) | < \tfrac{1}{2}h,$$

that is, $\quad f_n(x_1) > f(x_1) - \tfrac{1}{2}h \quad$ and $\quad f_n(x_2) < f(x_2) + \tfrac{1}{2}h,$

and therefore $\quad f_n(x_1) - f_n(x_2) > 0.$

But by hypothesis $f_n(x)$ is monotonic nondecreasing, *i. e.*,

$$f_n(x_1) - f_n(x_2) \leqq 0.$$

We have then reached a contradiction, and therefore the hypothesis that $f(x)$ is not a monotonic nondecreasing function of x is invalid.

2. Since $f_n(x)$ and $f(x)$ are monotonic nondecreasing functions, their variations over any interval $x_1, x_2 (a \leqq x_1 < x_2 \leqq b)$ are the differences

$$f_n(x_2) - f_n(x_1) \quad \text{and} \quad f(x_2) - f(x_1).$$

Suppose $\quad V_n(x_1 x_2) = f_n(x_2) - f_n(x_1) \quad$ and $\quad V(x_1 x_2) = f(x_2) - f(x_1)$

Then $$\underset{n}{L} V_n(x_1 x_2) = V(x_1 x_2).$$

For, $$\underset{n}{L} V_n(x_1 x_2) = \underset{n}{L} (f_n(x_2) - f_n(x_1)) = f(x_2) - f(x_1) = V(x_1 x_2).$$

3. From this we deduce the following

Lemma: If $\underset{n}{L} f_n(x) = f(x)$ under the conditions imposed in Theorem A, then it is possible to find for every $\epsilon > 0$ and for every ξ, $(a \leqq \xi \leqq b)$, a $\delta_{\epsilon\xi}$ and an $n_{\epsilon\xi}$ such that, if

$$|x - \xi| \leqq \delta_{\epsilon\xi} \quad \text{and} \quad n \geqq n_{\epsilon\xi}$$

we have

$$|f_n(x) - f(x)| \leqq \epsilon.$$

For, $f(x)$ is continuous, that is, for every $\epsilon > 0$ we can find a $\delta_{\epsilon\xi}$, such that, if $|x - \xi| \leqq \delta_{\epsilon\xi}$ we have

$$|f(\xi) - f(x)| \leqq \tfrac{1}{6}\epsilon.$$

Hence $$V(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) \leqq \tfrac{2}{6}\epsilon.$$

Now by 2, $$\underset{n}{L}\, V_n(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) = V(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi})\,;$$

that is, for every $\epsilon > 0$ there exists an $n'_{\epsilon\xi}$ such that $n \geqq n'_{\epsilon\xi}$ implies:

$$|\, V_n(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) - V(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) \,| \leqq \tfrac{1}{6}\epsilon,$$

and therefore $$V_n(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) \leqq \tfrac{3}{6}\epsilon.$$

Moreover since

$$\underset{n}{L}\, f_n(\xi) = f(\xi)$$

for every $\epsilon > 0$ we can find an $n''_{\epsilon\xi}$ such that $n \geqq n''_{\epsilon\xi}$ implies:

$$|f_n(\xi) - f(\xi)| \leqq \tfrac{1}{6}\epsilon.$$

Now

$$|f_n(x) - f(x)| \leqq |f_n(x) - f_n(\xi)| + |f_n(\xi) - f(\xi)| + |f(\xi) - f(x)|.$$

If then we take

$$|x - \xi| \leqq \delta_{\epsilon\xi} \quad \text{and} \quad n \geqq n_{\epsilon\xi} \geqq \begin{cases} n'_{\epsilon\xi} \\ n''_{\epsilon\xi} \end{cases}$$

we have

$$|f_n(x) - f(x)| \leqq V_n(\xi - \delta_{\epsilon\xi}, \xi + \delta_{\epsilon\xi}) + \tfrac{1}{6}\epsilon + \tfrac{1}{6}\epsilon \leqq \epsilon.$$

4. We now proceed to the proof of Theorem A. Suppose there is given an $\epsilon > 0$. We can then find by our Lemma, for every point ξ on the interval $a \leqq \xi \leqq b$, a $\delta_{\epsilon\xi}$ and an $n_{\epsilon\xi}$ such that if x lies in the $\delta_{\epsilon\xi}$ vicinity of ξ and $n \geqq n_{\epsilon\xi}$ we have

$$|f_n(x) - f(x)| \leqq \epsilon.$$

The $\delta_{\epsilon\xi}$ vicinities will form an infinity of segments covering the interval $a \leqq x \leqq b$. Hence by the Heine-Borel Theorem,* it is possible to select a finite number of those vicinities which will cover the interval. To these vicinities there will correspond a finite number of $n_{\epsilon\xi}$ as determined by the Lemma. Of these $n_{\epsilon\xi}$ we choose n_ϵ, the largest. This n_ϵ is evidently independent of the point ξ. Then for every $n \geqq n_\epsilon$ and $a \leqq x \leqq b$ we shall have

$$|f_n(x) - f(x)| \leqq \epsilon,$$

that is, the convergence is uniform.

5. Denote by $V_n(x)$ and $V(x)$ the total variations (in the sense of Jordan) of $f_n(x)$ and $f(x)$ from a to x. Then we can state the following theorem :†

THEOREM B. *If* $\underset{n}{L} f_n(x) = f(x)$ *and* $\underset{n}{L} V_n(x) = V(x)$ *for every* x *on the interval* $a \leqq x \leqq b$, *and* $f(x)$ *is continuous, then the convergence of* $f_n(x)$ *to* $f(x)$ *is uniform for* $a \leqq x \leqq b$.

The proof of this theorem can be made on the same lines as the proof of Theorem A.

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* *Annales de l'Ecole Normale Superieure*, vol. 12, p. 51. Veblen and Lennes, *Infinitesimal Analysis*, p. 34.

† This theorem was suggested by Professor Osgood.